

# A REMARK ON IMAGINARY PART OF RESONANCE POINTS

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**ABSTRACT.** In this paper we prove for rank one perturbations that negative two times reciprocal of the imaginary part of resonance point is equal to the rate of change of the scattering phase as a function of the coupling constant, where the coupling constant is equal to the real part of the resonance point. This equality is in agreement with Breit-Wigner formula from quantum scattering theory.

For general relatively trace class perturbations, we also give a formula for the spectral shift function in terms of resonance points, non-real and real.

## 1. INTRODUCTION

Given a self-adjoint operator  $H_0$  and a relatively compact self-adjoint operator  $V$ , a resonance point  $r_z$  of the triple  $(z; H_0, V)$  is a pole of the meromorphic function

$$s \mapsto R_z(H_s)V,$$

where  $H_s = H_0 + sV$ . Under certain conditions on the pair  $(H_0, V)$ , which ensure existence of scattering theory (the limiting absorption principle), resonance points  $r_z$  have limit values  $r_{\lambda+i0}$  for a.e.  $\lambda \in \mathbb{R}$ . In case  $r_{\lambda+i0}$  is real, it has several interpretations, discussed in details in the introduction of [Az<sub>2</sub>]. One of the interpretations is that one of the scattering phases  $\theta_j(\lambda, r)$  of the scattering matrix

$$S(\lambda; H_r, H_0),$$

considered as an analytic function of the coupling constant  $r$ , suffers a sudden jump by an integer multiple of  $2\pi$  when  $r$  crosses the real resonance point  $r_\lambda$ . By a scattering phase  $\theta_j(\lambda, r)$  we mean this:  $e^{i\theta_j(\lambda, r)}$  is one of the eigenvalues of the scattering matrix  $S(\lambda; H_r, H_0)$ .

The limit values of resonance points  $r_{\lambda+i0}$  with non-zero imaginary parts have not been investigated. In this paper for rank-one perturbations, in which case there is only one non-zero scattering phase  $\theta_1(\lambda, r)$ , we prove the following formula:

$$(1.1) \quad \left. \frac{\partial \theta_1(\lambda, r)}{\partial r} \right|_{r=\operatorname{Re} r_{\lambda+i0}} = -\frac{2}{\operatorname{Im} r_{\lambda+i0}}, \quad \text{a.e. } \lambda.$$

Since  $r_{\lambda+i0}$  is a pole of the scattering matrix, this formula is in agreement with Breit-Wigner formula from quantum scattering theory, see e.g. [Bo, Chapter XVIII], [T, Chapter 13], with that difference that the phase is considered as a function of the coupling constant instead of energy.

## 2. PROOF

**Theorem 2.1.** *If  $V$  is a rank one self-adjoint operator and if  $H_0$  is a self-adjoint operator, acting on a separable complex Hilbert space  $\mathcal{H}$ , then for a.e.  $\lambda$  the formula (1.1) holds. In this*

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formula, for  $z$  outside the essential spectrum of  $H_0$ , the number  $r_z$  is a (unique) pole of the meromorphic function

$$\mathbb{C} \ni s \mapsto (H_0 + sV - z)^{-1}V,$$

and  $r_{\lambda+i0} = \lim_{y \rightarrow 0^+} r_{\lambda+iy}$ , and  $e^{i\theta_1(\lambda, r)}$  is the (unique) non-trivial eigenvalue of the scattering matrix  $S(\lambda; H_r, H_0)$ .

*Proof.* Before proceeding to the proof we make some remarks. Since  $V$  has rank one, the default premise of this theorem — the limiting absorption theorem — holds, see e.g. [Y, Chapter 6]; this is necessary for existence of the scattering matrix and therefore for that of the scattering phase. In the proof we use notation of the introduction of [Az<sub>2</sub>] and of [Az]. Objects such as fiber Hilbert space  $\mathfrak{h}_\lambda(H_0)$ , the wave matrices  $w_\pm(\lambda; H_r, H_0)$  and the scattering matrix  $S(\lambda; H_r, H_0)$  are defined by explicit formulas of the constructive approach to stationary scattering theory, given in [Az] and outlined in the introduction of [Az<sub>2</sub>]. Therefore, we can consider all these objects as well defined for all  $r$  except a discrete set, as long as  $\lambda$  belongs to a pre-defined set of full Lebesgue measure, which comes from the limiting absorption principle, for details see introduction of [Az<sub>2</sub>]. Since  $V$  has rank one, it can be easily shown that the resonance point  $r_z$  is a well-defined single-valued Herglotz function in the upper complex half-plane  $\mathbb{C}_+$  (to be precise,  $r_z$  is a Herglotz function if  $V$  is non-negative, otherwise  $-r_z$  is a Herglotz function).

In the following  $\lambda$  is a real number from a set  $\Lambda(H_0, F)$  of full Lebesgue measure (that is,  $\mathbb{R} \setminus \Lambda(H_0, F)$  is a set of measure zero). Here  $F$  is a *rigging operator* which is necessary for constructive approach to stationary scattering theory, see [Az], [Az<sub>2</sub>, Introduction]. In particular, the rigging operator  $F$  generates a rigging  $(\mathcal{H}_+, \mathcal{H}, \mathcal{H}_-)$  of the Hilbert space  $\mathcal{H}$  and it allows to define the evaluation operator  $\mathcal{E}_\lambda(H_r): \mathcal{H}_+ \rightarrow \mathfrak{h}_\lambda(H_r)$  for all non-resonant values of the coupling constant  $r$ . The diamond conjugate  $\mathcal{E}_\lambda^\diamond(H_r)$  of  $\mathcal{E}_\lambda(H_r)$  is an operator from  $\mathfrak{h}_\lambda(H_r)$  to  $\mathcal{H}_-$  such that for any  $f \in \mathcal{H}_+$  and any  $g \in \mathfrak{h}_\lambda(H_r)$  one has  $\langle \mathcal{E}_\lambda(H_r)f, g \rangle_{\mathfrak{h}_\lambda(H_r)} = \langle f, \mathcal{E}_\lambda^\diamond(H_r)g \rangle_{\mathcal{H}_+, \mathcal{H}_-}$ , where  $\langle \cdot, \cdot \rangle_{\mathcal{H}_+, \mathcal{H}_-}$  is the pairing of the rigging triple  $(\mathcal{H}_+, \mathcal{H}, \mathcal{H}_-)$ . Finally, the operator  $V$  can be treated as an operator from  $\mathcal{H}_-$  to  $\mathcal{H}_+$ . For details see [Az] and [Az<sub>2</sub>].

Now we proceed to the proof. Since  $\lambda$  is fixed, we write  $\theta_j(r)$  for  $\theta_j(\lambda, r)$ . We have the formula [Az, Theorem 7.3.3]

$$\frac{dS(\lambda; H_r, H_0)}{dr} = -2\pi i w_+(\lambda; H_0, H_r) \mathcal{E}_\lambda(H_r) V \mathcal{E}_\lambda^\diamond(H_r) w_+(\lambda; H_r, H_0) S(\lambda; H_r, H_0).$$

Since the operator  $S(\lambda; H_r, H_0)$ , acting on the fiber Hilbert space  $\mathfrak{h}_\lambda = \mathfrak{h}_\lambda(H_0)$ , is unitary, it follows that

$$\frac{dS(\lambda; H_r, H_0)}{dr} S^{-1}(\lambda; H_r, H_0) = -2\pi i w_+(\lambda; H_0, H_r) \mathcal{E}_\lambda(H_r) V \mathcal{E}_\lambda^\diamond(H_r) w_+(\lambda; H_r, H_0).$$

Further mostly we write  $S(r)$  for  $S(\lambda; H_r, H_0)$ . Since (see [Az<sub>2</sub>, Corollary 5.3.8])

$$w_+(\lambda; H_0, H_r) w_+(\lambda; H_r, H_0) = 1_{\mathfrak{h}_\lambda},$$

taking traces of both sides of the previous equality gives

$$\begin{aligned} \text{Tr}(S'(r)S^{-1}(r)) &= -2\pi i \text{Tr}_{\mathfrak{h}_\lambda} \left( \mathcal{E}_\lambda(H_r) V \mathcal{E}_\lambda^\diamond(H_r) \right) \\ &= -2\pi i \text{Tr}_{\mathcal{H}_+} \left( V \mathcal{E}_\lambda^\diamond(H_r) \mathcal{E}_\lambda(H_r) \right). \end{aligned}$$

The product  $\mathcal{E}_\lambda^\diamond(H_r)\mathcal{E}_\lambda(H_r)$  is equal to  $\frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_r)$  (as an operator from  $\mathcal{H}_+$  to  $\mathcal{H}_-$ , see [Az, (5.5)]). So,

$$\operatorname{Tr}(S'(r)S^{-1}(r)) = -2i \operatorname{Tr}_{\mathcal{H}_+}(V \operatorname{Im} R_{\lambda+i0}(H_r)).$$

This we can rewrite as (see [Az, §2.15])

$$\operatorname{Tr}(S'(r)S^{-1}(r)) = -2i \lim_{y \rightarrow 0^+} \operatorname{Tr}_{\mathcal{H}}(V \operatorname{Im} R_{\lambda+iy}(H_r)).$$

Hence,

$$\operatorname{Tr}(S'(r)S^{-1}(r)) = - \lim_{y \rightarrow 0^+} \operatorname{Tr}_{\mathcal{H}}(V R_{\lambda+iy}(H_r) - V R_{\lambda-iy}(H_r)).$$

Now we recall that a resonance point  $r_z$  corresponding to a triple  $(z; H_0, V)$  is a complex number such that  $(s - r_z)^{-1}$  is an eigenvalue of the compact operator  $R_z(H_s)V$  (see [Az2, §3.1], more specifically [Az2, (3.1.2)]). Since by the premise the operator  $V$  has rank 1, the operator  $R_z(H_s)V$  has only one eigenvalue. Therefore, (using  $\bar{r}_z = r_{\bar{z}}$ , which is obvious)

$$\begin{aligned} \operatorname{Tr}(S'(r)S^{-1}(r)) &= - \lim_{y \rightarrow 0^+} ((r - r_z)^{-1} - (r - \bar{r}_z)^{-1}) \\ &= - \lim_{y \rightarrow 0^+} \frac{-\bar{r}_z + r_z}{(r - r_z)(r - \bar{r}_z)} \\ &= - \lim_{y \rightarrow 0^+} \frac{2i \operatorname{Im} r_z}{(r - r_z)(r - \bar{r}_z)}. \end{aligned}$$

Taking the limit and replacing  $r$  by  $\operatorname{Re} r_{\lambda+i0}$ , we get

$$\begin{aligned} \operatorname{Tr}(S'(r)S^{-1}(r))|_{r=\operatorname{Re} r_{\lambda+i0}} &= - \frac{2i \operatorname{Im} r_{\lambda+i0}}{(\operatorname{Re} r_{\lambda+i0} - r_{\lambda+i0})(\operatorname{Re} r_{\lambda+i0} - \bar{r}_{\lambda+i0})} \\ &= -2i \frac{\operatorname{Im} r_{\lambda+i0}}{|\operatorname{Im} r_{\lambda+i0}|^2}. \end{aligned}$$

On the other hand, since  $\operatorname{rank}(V) = 1$ , the fiber Hilbert space  $\mathfrak{h}_\lambda(H_0)$  on which the scattering matrix  $S(r)$  acts is one-dimensional. Hence,  $S(\lambda; H_r, H_0)$  is the operator of multiplication by its eigenvalue:

$$S(\lambda; H_r, H_0) = e^{i\theta_1(r)} \cdot 1_{\mathfrak{h}_\lambda(H_0)}.$$

It follows that

$$\begin{aligned} \operatorname{Tr}(S'(r)S^{-1}(r))|_{r=\operatorname{Re} r_{\lambda+i0}} &= \frac{de^{i\theta_1(r)}}{dr} e^{-i\theta_1(r)}|_{r=\operatorname{Re} r_{\lambda+i0}} \\ &= i\theta_1'(\operatorname{Re} r_{\lambda+i0}). \end{aligned}$$

Comparing the last two formulas completes the proof.  $\square$

In this proof the following equality is derived:

$$\operatorname{Tr}(S'(r)S^{-1}(r)) = - \frac{2i \operatorname{Im} r_{\lambda+i0}}{(r - r_{\lambda+i0})(r - \bar{r}_{\lambda+i0})}.$$

This equality can be written as

$$\theta_1'(\lambda, r) = - \frac{2\beta}{|r - \alpha|^2 + |\beta|^2},$$

where  $\alpha = \operatorname{Re} r_{\lambda+i0}$ , and  $\beta = \operatorname{Im} r_{\lambda+i0}$ . This equality implies that for any  $a < b$ , the eigenvalue  $e^{i\theta_1(\lambda; b, a)}$  of the scattering matrix  $S(\lambda; H_b, H_a)$  is not equal to 1; otherwise we get a contradiction with Rolle's theorem. Since

$$\int_{-\infty}^{\infty} \frac{2\beta}{|r - \alpha|^2 + |\beta|^2} = 2\pi,$$

it follows that for a.e.  $\lambda$  with  $\beta \neq 0$

$$\theta_1(\lambda; +\infty, -\infty) = -2\pi.$$

Since for a.e.  $\lambda$  (modified Pushnitski's formula, see [Az, Theorem 9.2.2])

$$\xi^{(a)}(\lambda; H_1, H_0) = -\frac{1}{2\pi}\theta_1(\lambda; 1, 0),$$

for a.e.  $\lambda$  with  $\beta \neq 0$  this gives (Yavrjan's formula)

$$\xi^{(a)}(\lambda; H_\infty, H_{-\infty}) = 1.$$

Let  $V = F^* J F$ , such that  $FR_z(H_0)F^*$  is trace class. This condition implies the limiting absorption principle, that is, existence of the uniform limit  $FR_{\lambda+i0}(H_s)F^*$ . This allows to carry out the constructive approach to stationary scattering theory, in particular, to show that the operator  $S(\lambda; H_r, H_0) - 1$  is trace class for all non-resonant  $r$ , cf. [Az<sub>2</sub>, Introduction], [AzD] or [Az<sub>4</sub>], for trace class  $V$  see [Az].

Given these conditions, following the same argument one can prove the following equality:

$$(2.1) \quad \sum_{j=1}^{\infty} \theta_j(\lambda; b, a) = - \int_a^b \sum_{j=1}^{\infty} \frac{2\beta_j}{|r - \alpha_j|^2 + |\beta_j|^2} dr,$$

where  $r_{\lambda+i0}^j = \alpha_j + i\beta_j$  is  $j$ th resonance point with non-zero  $\beta_j$ , that is,  $(s - r_{\lambda+i0}^j)^{-1}$  is  $j$ th eigenvalue of  $JFR_{\lambda+i0}(H_s)F^*$ . Combining this with modified Pushnitski's formula (for trace class  $V$ , [Az, Theorem 9.2.2])

$$\xi^{(a)}(\lambda; H_b, H_a) = -\frac{1}{2\pi} \sum_{j=1}^{\infty} \theta_j(\lambda; b, a),$$

gives

$$\xi^{(a)}(\lambda; H_b, H_a) = \frac{1}{2\pi} \int_a^b \sum_{j=1}^{\infty} \frac{2\beta_j}{|r - \alpha_j|^2 + |\beta_j|^2} dr.$$

To obtain a formula for SSF  $\xi$  one has to add to this integral the sum of resonance indices of real resonance points  $r_{\lambda+i0}^j$  which belong to the interval  $[a, b]$ , that is, the singular spectral shift function  $\xi^{(s)}(\lambda; H_b, H_a)$ :

$$\xi(\lambda; H_b, H_a) = \frac{1}{2\pi} \int_a^b \sum_{j=1}^{\infty} \frac{2\beta_j}{|r - \alpha_j|^2 + |\beta_j|^2} dr + \sum_{r_\lambda \in [a, b]} \text{ind}_{res}(\lambda; H_{r_\lambda}, V).$$

That is, resonance points  $r_{\lambda+i0}^j$  with zero imaginary part  $\beta_j$  still contribute to the spectral shift function in the form of resonance index.

The formula (2.1) raises a question: is there a natural one-to-one correspondence between scattering phases  $\theta_j(\lambda, r)$  and resonance points  $r_{\lambda+i0}^j$ ? The answer to this question is positive: resonance points  $r_z^j$  are poles of the analytic continuation of the scattering matrix  $S(z; H_s, H_0)$  considered as a meromorphic function of  $s$ , and at a pole the analytic continuation of one of the eigenvalues  $e^{i\theta_j(z, s)}$  of the scattering matrix goes either to  $\infty$  or  $0$ , see [Az<sub>3</sub>]. This gives the required one-to-one correspondence. At the same time, for  $V$  with rank  $> 1$ , individual terms on both sides of the formula (2.1) are not equal.

## REFERENCES

- [Az] N. A. Azamov, *Absolutely continuous and singular spectral shift functions*, Dissertationes Math. **480** (2011), 1–102.
- [Az<sub>2</sub>] N. A. Azamov, *Spectral flow inside essential spectrum*, arXiv 1407.3551v2, to appear in Dissertationes Math.
- [Az<sub>3</sub>] N. A. Azamov, *Resonance index and singular  $\mu$ -invariant*, arXiv 1501.07673.
- [Az<sub>4</sub>] N. A. Azamov, *A constructive approach to stationary scattering theory*, arXiv math-ph 1302.4142.
- [AzD] N. A. Azamov, Th. Daniels, *Singular spectral shift function for Schrödinger operators*, in preparation.
- [Bo] A. Böhm, *Quantum mechanics*, Springer-Verlag, New York, Heidelberg, Berlin.
- [T] J. R. Taylor, *Scattering theory*, John Wiley & Sons, Inc. New York.
- [Y] D. R. Yafaev, *Mathematical scattering theory: general theory*, Providence, R. I., AMS, 1992.

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